

# CONVENIENT DESCRIPTIONS OF WEIGHT FUNCTIONS IN TIME-FREQUENCY ANALYSIS

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**ABSTRACT.** Let  $v$  be a submultiplicative weight. Then we prove that  $v$  satisfies GRS-condition, if and only if  $v \cdot e^{-\varepsilon|\cdot|}$  is bounded for every positive  $\varepsilon$ . We use this equivalence to establish identification properties between weighted Lebesgue spaces, and between certain modulation spaces and Gelfand-Shilov spaces.

## 0. INTRODUCTION

In [4], Gel'fand, Raikov and Shilov considered a family of matrix classes in  $\mathbb{Z}^d \times \mathbb{Z}^d$ , where each matrix class,  $\mathcal{A}_v$ , depends on a weight function  $v$  on  $\mathbb{R}^d$ , and consists of all matrices  $A = (a(j, k))_{j, k \in \mathbb{Z}^d}$  such that

$$\sup_{j \in \mathbb{Z}^d} a(j, j - k)v(k) \in \ell^1.$$

Here the weight  $v$  is positive and submultiplicative, i. e.  $v$  is even and fulfills  $v(x + y) \leq v(x)v(y)$ .

Since [1, 4], several investigations have been performed which confirm the importance of the class  $\mathcal{A}_v$ . For example if  $v \geq c$  for some positive constant  $c$ , then  $\mathcal{A}_v \subseteq \mathcal{B}(\ell^2(\mathbb{Z}^d))$ , the set of all matrices which are bounded on  $\ell^2(\mathbb{Z}^d)$ . Furthermore, in [6, 7], strong links between  $\mathcal{A}_v$  and certain modulation spaces are described, which appear in natural ways when considering non-stationary filters in signal analysis (cf. [14] and the references therein).

In [1, 4], the condition

$$\lim_{\ell \rightarrow \infty} v(\ell x)^{1/\ell} = 1, \quad x \in \mathbb{R}^d, \quad (1)$$

is introduced for weights  $v$  on  $\mathbb{R}^d$ , and link this condition to fundamental questions on inverse closed properties for matrix algebras. More precisely, for any submultiplicative weight  $v$  which fulfills (1), it is proved in [4] that the matrix class  $\mathcal{A}_v$  is so-called *inverse closed* on  $\ell^2(\mathbb{Z}^d)$ . That is,  $\mathcal{A}_v$  is contained in  $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ , and if  $A \in \mathcal{A}_v$  is invertible on  $\ell^2(\mathbb{Z}^d)$ , then its inverse  $A^{-1}$ , does also belong to  $\mathcal{A}_v$ . Furthermore, if  $v$  is submultiplicative and inverse closed on  $\ell^2(\mathbb{Z}^d)$ , then it is necessary that  $v$  satisfies (1) (cf. [9, Corollary 5.31]).

The condition (1) (the so-called *GRS-condition* or *Gel'fand-Raikov-Shilov condition*) is therefore strongly linked to inverse closed properties for matrix algebras, and a submultiplicative weight which satisfies the GRS-conditions is called a *GRS weight*.

Later on, similar links between the GRS-conditions and inverse closed properties for pseudo-differential operators with symbols in certain modulation spaces were established (see e. g. [7]).

In [8], it is proved that if  $v$  is submultiplicative, then there are positive constants  $c$  and  $C$  such that

$$v(x) \leq Ce^{c|x|},$$

and it is easily seen that the right-hand side is a submultiplicative weight. We note however that the weight

$$v_c(x) = v_{c,1}(x) \equiv e^{c|x|} \quad (2)$$

does not satisfy the GRS-condition. Consequently, the matrix algebra  $\mathcal{A}_v$  is not inverse closed when  $v = v_c$  and  $c > 0$ . On the other hand, it is easily seen that the set of submultiplicative weights satisfying the GRS-condition is a monoid and contains the weights

$$x \mapsto (1 + |x|)^r \quad \text{and} \quad x \mapsto v_{c,s}(x) \equiv e^{c|x|^s}$$

when  $r \geq 0$ ,  $c \geq 0$  and  $0 \leq s < 1$ .

Since the (non-GRS) weight  $v_{c,1}$  is rather similar to the GRS weight  $v_{c,s}$  when  $s < 1$  is close to 1, it might be anticipated that the algebra  $\mathcal{A}_{v_{c,1}}$  should possess a weaker form of inverse closed property. Such property is established in [10], where it is proved that if  $A \in \mathcal{A}_{v_{c_1,1}}$  for some  $c_1 > 0$  is invertible on  $\ell^2(\mathbb{Z}^d)$ , then its inverse  $A^{-1}$  belongs to  $\mathcal{A}_{v_{c_2,1}}$ , for some  $c_2$ .

In this context we remark that in [3], a condition on sequences of weights, which extends the notion of GRS-condition and includes exponential type weights is considered. Furthermore, certain type of inverse closed property is established for matrix algebras which are parameterized with such sequences of weights. In particular, the classical equivalence between the inverse closed property of  $\mathcal{A}_v$  and the fact that  $v$  should be a GRS weight becomes a special case of [3, Theorem 2.1].

In this paper we establish a more narrow link between the GRS-condition and weights bounded by exponentials. More precisely, if  $v$  is submultiplicative, then we prove that  $v$  fulfills the GRS-conditions, if and only if for every  $\varepsilon > 0$ , there is a constant  $C_\varepsilon$  such that  $v \leq C_\varepsilon v_{\varepsilon,1}$ .

We also combine this result with Theorem 3.9 in [15] to establish relations with modulation spaces parameterized with weights satisfying GRS-conditions, and Gelfand-Shilov spaces.

# 1. IDENTIFICATIONS OF THE GRS-CONDITION IN TERMS OF EXPONENTIALS

We recall that a weight on  $\mathbb{R}^d$  is a positive function on  $\mathbb{R}^d$  which belongs to  $L_{loc}^\infty(\mathbb{R}^d)$ . The weight  $v$  on  $\mathbb{R}^d$  is submultiplicative, if  $v$  is positive and even, and satisfies

$$v(x+y) \leq v(x)v(y), \quad \text{for } x, y \in \mathbb{R}^d.$$

It is well-known that

$$v(x) \lesssim e^{c|x|}$$

for some  $c > 0$ , when  $v$  is submultiplicative. Furthermore, the weight  $v$  satisfies the *GRS-condition* whenever

$$\lim_{\ell \rightarrow \infty} v(\ell x)^{1/\ell} = 1, \quad \text{when } x \in \mathbb{R}^d, \quad (3)$$

holds.

We have now the following observation concerning submultiplicative weights.

**Proposition 1.** *Let  $v$  be a submultiplicative weight on  $\mathbb{R}^d$ . Then the following conditions are equivalent:*

- (1)  *$v$  satisfies the GRS-condition;*
- (2)  *$v$  satisfies  $v(x) \lesssim e^{\varepsilon|x|}$ , for every  $\varepsilon > 0$ .*

*Proof.* First assume that (2) holds. Let  $\varepsilon > 0$  and  $x \in \mathbb{R}^d$  be arbitrary and fixed. As

$$v(nx) \leq C e^{\varepsilon n|x|},$$

we have

$$v(nx)^{1/n} \leq C^{1/n} e^{\varepsilon|x|},$$

and since  $\varepsilon$  is arbitrary we have

$$\limsup_{n \rightarrow \infty} v(nx)^{1/n} \leq 1.$$

On the other hand, since  $v(x) \geq c$  for some  $c > 0$ , it follows that

$$\liminf_{n \rightarrow \infty} v(nx)^{1/n} \geq 1.$$

A combination of these estimates gives that  $\lim_{n \rightarrow \infty} v(nx)^{1/n}$  exists and is equal to 1, i. e. (1) holds.

Next we assume that (2) does not hold. Let  $\{e_1, \dots, e_d\}$  be the canonical basis in  $\mathbb{R}^d$ . Since  $v$  is submultiplicative,

$$v(x_1, \dots, x_d) \leq \prod_{j=1}^d v(x_j e_j).$$

Hence, if (2) is not fulfilled, then there are  $j \in \{1, \dots, d\}$  and  $\varepsilon > 0$  such that  $v(te_j)e^{-\varepsilon|t|}$  is unbounded. Without loss of generality we may assume that  $j = 1$ . The weight on  $\mathbb{R}$  defined by

$$u(t) := v(te_1)$$

is submultiplicative. Since  $u(t)e^{-\varepsilon t}$  is unbounded in  $[0, \infty[$ , we may find a sequence  $(t_n)$  such that  $0 < t_n + 1 < t_{n+1} \rightarrow \infty$  and

$$u(t_n) > e^{\varepsilon t_n}.$$

Let  $k_n$  be the integer part of  $t_n$ . As  $t_n < k_n + 1$  and  $u$  is submultiplicative,

$$u(t_n) \leq Cu(k_n + 1) \leq Cu(1)u(k_n),$$

where  $C = \max_{|\xi| \leq 1} u(\xi)$ . Hence

$$u(k_n) \geq \frac{u(t_n)}{Cu(1)} \geq \frac{e^{\varepsilon t_n}}{Cu(1)}$$

and

$$u(k_n)^{1/k_n} \geq \left( \frac{u(t_n)}{Cu(1)} \right)^{1/t_n} \geq \frac{e^\varepsilon}{(Cu(1))^{1/t_n}}.$$

Consequently

$$\liminf_{n \rightarrow \infty} u(k_n)^{1/k_n} \geq e^\varepsilon.$$

This implies that  $u$ , and therefore  $v$ , does not satisfy the GRS-condition. The proof is complete.  $\square$

The previous result means that  $\mathcal{A}_v$  is inverse closed in  $\mathcal{B}(\ell^2(\mathbb{Z}^d))$  if, and only if, it contains the Jaffard algebra of all matrices with exponential off-diagonal decay.

**Corollary 2.** *Let  $\Omega$  be the set of all weights on  $\mathbb{R}^{2d}$  which satisfy the GRS-condition. Then*

$$\bigcup_{\varepsilon > 0} \ell_{(v_\varepsilon)}^1 = \bigcap_{v \in \Omega} \ell_{(v)}^1 \quad \text{and} \quad \bigcup_{\varepsilon > 0} \mathcal{A}_{v_\varepsilon} = \bigcap_{v \in \Omega} \mathcal{A}_v. \quad (4)$$

*Proof.* This is a consequence of Proposition 1 and [3, Theorem 2.1].  $\square$

Next we analyze a condition on sequences of weights considered in [3].

**Proposition 3.** *Let  $\mathcal{W} = \{w_n\}_{n \in \mathbb{N}}$  be a decreasing sequence of submultiplicative weights. Then the following conditions are equivalent:*

- (1)  $\inf_n \lim_{\ell \rightarrow \infty} w_n(\ell x)^{1/\ell} = 1$  for every  $x \in \mathbb{R}^d$ .
- (2) for every  $\varepsilon > 0$ , there exists an  $n \in \mathbb{N}$  such that  $w_n(x)e^{-\varepsilon|x|}$  is bounded.

*Proof.* (2)  $\Rightarrow$  (1) is clear. Assume that (1) holds, and let  $V$  be the family of all submultiplicative and weights  $v : \mathbb{Z}^d \rightarrow \mathbb{R}_+$  such that

$$\sup_k \frac{v(k)}{w_n(k)} < \infty \quad \forall n \in \mathbb{N}.$$

By [2, Theorem 2.3] we have

$$\bigcap_{v \in V} \ell_{(v)}^1 = \bigcup_n \ell_{(w_n)}^1$$

algebraic and topologically (see also the proof of [3, Theorem 2.1]). Since every weight  $v \in V$  satisfies the GRS-condition, we have  $v = O(v_\varepsilon)$  for every  $\varepsilon > 0$ . Consequently,

$$\ell_{(v_\varepsilon)}^1 \subset \bigcup_n \ell_{(w_n)}^1$$

with continuous inclusion. The result now follows from Gröthendieck's factorization theorem (see [12, 1.2.20]).  $\square$

We finish the paper by applying our previous results to establish certain links between modulation spaces and Gelfand-Shilov spaces. We recall the definition of these spaces.

Let  $0 < h, s \in \mathbb{R}$  be fixed, and let  $\mathcal{S}_{s,h}(\mathbb{R}^d)$  be the set of all  $f \in C^\infty(\mathbb{R}^d)$  such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha|+|\beta|} \alpha!^s \beta!^s}$$

is finite. Here the supremum should be taken over all  $\alpha, \beta \in \mathbb{N}_0^d$  and  $x \in \mathbb{R}^d$ . We note that  $\mathcal{S}_{s,h}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$ , increases with  $h$  and  $s$ , where  $\mathcal{S}(\mathbb{R}^d)$  is the set of all Schwartz functions on  $\mathbb{R}^d$ . The *Gelfand-Shilov space*  $\mathcal{S}_s(\mathbb{R}^d)$  is the inductive limit of  $\mathcal{S}_{s,h}(\mathbb{R}^d)$  and the *Gelfand-Shilov distribution space*  $\mathcal{S}'_s(\mathbb{R}^d)$  is the projective limit of  $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ . In particular,

$$\mathcal{S}_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d), \quad \text{and} \quad \mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d). \quad (5)$$

Note that  $\mathcal{S}'_s(\mathbb{R}^d)$  is the dual of  $\mathcal{S}_s(\mathbb{R}^d)$  ([5, 11, 13]).

Evidently,  $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}_s(\mathbb{R}^d)$  increases with  $s$ , and  $\mathcal{S}'(\mathbb{R}^d) \subseteq \mathcal{S}'_s(\mathbb{R}^d)$  decreases with  $s$ . If  $s < 1/2$ , then  $\mathcal{S}_s(\mathbb{R}^d)$  is trivial.

The Fourier transform on  $\mathcal{S}'(\mathbb{R}^d)$  is the linear and continuous map which takes the form

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when  $f \in L^1(\mathbb{R}^d)$ . For every  $s \geq 1/2$ , the Fourier transform is continuous and bijective on  $\mathcal{S}_s(\mathbb{R}^d)$ , and extends uniquely to a continuous and bijective map on  $\mathcal{S}'_s(\mathbb{R}^d)$ .

Next we recall the definition of modulation spaces. Let  $1/2 \leq s_0 < s$ , and let  $\phi \in \mathcal{S}_{s_0}(\mathbb{R}^d) \setminus 0$  be fixed. Then the *short-time Fourier transform*  $V_\phi f$  of  $f \in \mathcal{S}'_s(\mathbb{R}^d)$  is the element in  $\mathcal{S}'_s(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})$ , defined by the formula

$$(V_\phi f)(x, \xi) := \mathcal{F}(f \cdot \phi(\cdot - x))(\xi).$$

If in addition  $f$  is in the Schwartz class, then  $V_\phi f$  is given by

$$(V_\phi f)(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\phi(y-x)} e^{-i\langle y, \xi \rangle} dy.$$

Let  $v$  be a submultiplicative weight on  $\mathbb{R}^d$ . Then the weight  $\omega$  on  $\mathbb{R}^d$  is called  $v$ -moderate if

$$\omega(x+y) \leq C\omega(x)v(y),$$

for some constant  $C$  which is independent of  $x \in \mathbb{R}^d$  and  $y \in \mathbb{R}^d$ .

Let  $\phi \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ ,  $1/2 < s < 1$ ,  $p, q \in (0, \infty]$ ,  $v$  be submultiplicative on  $\mathbb{R}^{2d}$ , and let  $\omega$  be a  $v$ -moderate weight on  $\mathbb{R}^{2d}$ . Then the *modulation space*  $M_{(\omega)}^{p,q}(\mathbb{R}^d)$  is the quasi-Banach space which consists of all  $f \in \mathcal{S}'_{1/2}(\mathbb{R}^d)$  such that

$$\|f\|_{M_{(\omega)}^{p,q}} := \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi) \omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} \quad (6)$$

is finite (with obvious modifications when  $p = \infty$  or  $q = \infty$ ). The definition of  $M_{(\omega)}^{p,q}(\mathbb{R}^d)$  is independent of the choice of  $\phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus 0$ , and different  $\phi$  gives rise to equivalent norms. (See e. g. [6]).

According to [6, Prop. 11.3.1],

$$\bigcap_{s \geq 0} M_{(w_s)}^\infty(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d),$$

where  $w_s(x) = (1 + |x|)^s$ . However, when considering all the weights satisfying the GRS-condition, the intersection is no longer a Fréchet space but an inductive limit of Banach spaces.

**Proposition 4.** *Let  $p, q \in (0, \infty]$ , and let  $\Omega$  be the set of all submultiplicative weights on  $\mathbb{R}^{2d}$  which satisfy the GRS-condition.. Then*

$$\bigcap_{v \in \Omega} M_{(v)}^{p,q}(\mathbb{R}^d) = \mathcal{S}_1(\mathbb{R}^d) \quad \text{and} \quad \bigcup_{v \in \Omega} M_{(1/v)}^{p,q}(\mathbb{R}^d) = \mathcal{S}'_1(\mathbb{R}^d). \quad (7)$$

For the proof we note that if  $\omega$  is a moderate weight on  $\mathbb{R}^{2d}$ ,  $p \in (0, \infty]$  and  $f \in \mathcal{S}'_1(\mathbb{R}^d)$ , then  $f \in M_{(\omega)}^{p,p}(\mathbb{R}^d)$ , if and only if

$$\left\{ \left( \iint_{n+Q} |V_\phi f(x, \xi)|^p dx d\xi \right)^{1/p} \right\}_{n \in \mathbb{Z}^{2d}} \in \ell_{(\omega)}^p(\mathbb{Z}^{2d}). \quad (8)$$

Here  $Q$  is the cube  $[0, 1]^{2d}$ .

*Proof.* By well-known embedding properties for modulation spaces, it suffices to prove the result in the case  $p = q = 1$  or  $p = q = \infty$  (cf. [6]). Let  $\phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\}$  be fixed and let  $Q := [0, 1]^{2d}$  be the unit cube. For any submultiplicative weight  $v$  on  $\mathbb{R}^{2d}$  and  $f \in \mathcal{S}'_1(\mathbb{R}^d)$  it turns out that  $f \in M_{(v)}^1(\mathbb{R}^d)$ , if and only if (8) holds for  $p = 1$ . Consequently, the first identity in (4) gives

$$\bigcap_{v \in \Omega} M_{(v)}^1(\mathbb{R}^d) = \bigcup_{\epsilon > 0} M_{(v_\epsilon)}^1(\mathbb{R}^d) = \mathcal{S}_1(\mathbb{R}^d),$$

where the last identity follows from Theorem 3.9 in [15]. The second statement of the proposition follows from the fact that the dual of  $M_{(v)}^1$  is given by  $M_{(1/v)}^\infty$  (see e. g. [6]).  $\square$

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